CONVEX POWER FLOW MODELS FOR SCALABLE ELECTRICITY MARKET MODELLING

Frederik GETH
EnergyVille & KU Leuven, Belgium
frederik.geth@energyville.be

Reinhilde D'HULST
EnergyVille & VITO, Belgium
reinhilde.dhulst@energyville.be

Dirk VAN HERTEM
EnergyVille & KU Leuven, Belgium
dirk.vanhertem@energyville.be

ABSTRACT

Optimal power flow problems and market clearing approaches are converging: cost-optimal scheduling of loads and generators should be performed while taking the grid's physics and operational envelopes into account. Within the SmartNet project, the idea is to consider the grid's physical behaviour in market clearing approaches. Taking the physics of power flow into account, while managing solution times, demands pragmatic approaches. Convex relaxation and linear approximation are two such approaches to manage computational tractability. This work gives an overview of recent OPF formulations, and their relaxations and approximations. The hierarchy of the approaches is detailed, as well as the loss of properties resulting from the relaxation process.

INTRODUCTION

Optimal power flow (OPF) on the one side and unit commitment, economic dispatch and market clearing on the other side are converging: cost-optimal scheduling of loads and generators should be performed while taking the grid's physics and operational envelopes into account. This convergence of OPF and economic dispatch will only increase since generators connected to the distribution system are getting more and more involved in the operational planning of the power system. Consequently, a decoupling of the grid's physics from economic dispatch calculations will increasingly lead to infeasible solutions. At the same time, both the mathematical modelling and the increasing capabilities of computational tools has made the combination of both OPF and market clearing possible.

Historically, due to limited computational resources, unit scheduling was typically performed using copper plate or network flow models. To avoid problems due to the limited accuracy of such models, a post-clearing AC power flow check is performed and redispatch is performed in case of expected operational difficulties. Such actions may lead to inefficiency. Including a more accurate network model in the market clearing will help to avoid countertrading and its associated costs.

Taking the physics of power flow into account, while guaranteeing limited solution times, demands pragmatic approaches. Convex relaxation and linear approximation are two such approaches to manage computational tractability, which will be detailed throughout this article.

The Smartnet Project

The SmartNet project [1] aims at providing solutions to clarify architectures for optimized interaction between TSOs and DSOs. A market architecture for ancillary services is envisioned in which the grid limitation constraints (of both DSOs and TSOs) are incorporated. Therefore, in the SmartNet project, a market clearing methodology is developed respecting the power system's physical limits.

Optimal Power Flow

In general, optimal power flow (OPF) encompasses any power system optimisation problem with physical power flow as part of the model (i.e. as part of the constraints). This does not exclude much, therefore OPF in general can be: multiperiod, security-constrained, DC or AC, or based on approximation (e.g. linearized 'DC' OPF). Market clearing algorithms respecting the power flow equations are also OPF problems.

Paper scope and structure

This article will not provide a detailed discussion of OPF formulation properties, but aims to map the landscape of recent OPF formulations and their applications. Only balanced power flow formulations are considered.

First, OPF is framed in the context of mathematical optimisation. Next, the OPF formulations are discussed and compared. Finally, the paper concludes.

OPF REFERENCE MODELS

Within optimal power flow models, the power flow equations are considered as equality constraints to an optimization problem.

Power flow is generally considered as a complex-valued problem, with j the imaginary unit, $\angle$ the complex angle and * the complex conjugate operator. The complex power flow from i to j is S_{ij}; the complex voltage at node i is U_i.

$$S_{ij} = P_{ij} + jQ_{ij}$$
$$U_i = |U_i|\angle\theta_i = |U_i|(cos\theta_i + jsin\theta_i) = e_i + jf_i$$

The power flow equations are commonly derived for pi-sections with a series $(y_{l,s} = g_{l,s} + b_{l,s})$ and shunt admittance $(y_{ij,\text{sh}} = g_{ij,\text{sh}} + jb_{ij,\text{sh}})$, see Figure 1.

Two formulations of the power flow equations have been used widely: a quadratic one, based on the rectangular-complex notation, and a polar-complex one, using sine

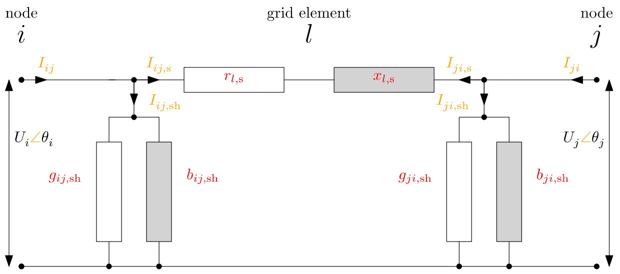

Figure 1: Pi-representation of a grid element.

and cosine functions.

$$S_{ij} = P_{ij} + jQ_{ij} = |U_i|^2 y^*_{ij,\text{sh}} + U_i\left(U_i^* - U_j^*\right) y^*_{l,s}$$

$$P_{ij} = \left(g_{ij,\text{sh}} + g_{l,s}\right)|U_i|^2 - g_{l,s}|U_i|\left|U_j\right| cos\left(\theta_i - \theta_j\right) - b_{l,s}|U_i|\left|U_j\right| sin\left(\theta_i - \theta_j\right)$$

$$Q_{ij} = -\left(b_{ij,\text{sh}} + b_{l,s}\right)|U_i|^2 + b_{l,s}|U_i|\left|U_j\right| cos\left(\theta_i - \theta_j\right) - g_{l,s}|U_i|\left|U_j\right| sin\left(\theta_i - \theta_j\right)$$

An alternative formulation was developed by Baran and Wu in 1989 [2], which simplified power flow modelling in radial grids (voltage angle variables are substituted out). This formulation of the power flow equations is commonly referred to as DistFlow.

$$P_{ij} + jQ_{ij} + P_{ji} + jQ_{ji}$$

$$P^{\text{loss}}_{ij,s} + P^{\text{loss,}}_{ij,\text{sh}} + P^{\text{loss,}}_{ji,\text{sh}} + j\left(Q^{\text{loss}}_{ij,s} + Q^{\text{loss}}_{ij,\text{sh}} + Q^{\text{loss,}}_{ji,\text{sh}}\right)$$

$$P^{\text{loss}}_{ij,s} + jQ^{\text{loss}}_{ij,s} = z_{l,s}\left|I_{ij,s}\right|^2$$

$$P^{\text{loss}}_{ij,\text{sh}} + jQ^{\text{loss}}_{ij,\text{sh}} = y_{ij,\text{sh}}|U_i|^2$$

$$\left|U_j\right|^2 = |U_i|^2 - 2\left(r_{l,s}P_{ij,s} + x_{l,s}Q_{ij,s}\right) + \left(r^2_{l,s} + x^2_{l,s}\right)\left|I_{ij,s}\right|^2$$

Furthermore, an OPF typically includes a number of operational line bounds (current, voltage, power).

Optimisation classes

Since quite a while it is understood that the true distinction between easy-to-solve and hard-to-solve problems aligns with convex versus nonconvex optimisation. In theory and in practice, large convex problems can be solved reliably (convergence guaranteed), quickly (polynomial-time) and to global optimality. With nonconvex (smooth) optimisation, one largely has to choose between solving problems quickly but locally optimal or globally optimal but slowly.

The optimisation class hierarchy is:

LP ⊂ QP ⊂ SOCP ⊂ SDP ⊂ NCQCP ⊂ NLP [1]

Convex relaxation

Any nonconvex quadratically constrained optimisation problem (NCQCP) can be reformulated as a semidefinite programming (SDP) problem with the addition of a single nonconvex *rank-1* constraint. After removal of the rank constraint a SDP problem remains [3], which can be solved with SDP solvers. This step of removing nonconvex constraints is referred to as the 'convex relaxation' step.

Relaxations, by a process of only removing equations from the feasible set of an original problem, provide strong quality guarantees on the solution of both problems:

- if the original problem is feasible, the relaxed problem is feasible;
- if the relaxed problem is infeasible, the original problem is infeasible;
- the objective of the relaxed problem will be a lower bound (minimisation) for the objective of the original problem.

Approximations (modifications of constraints which cannot be shown to be relaxations), do not provide such guarantees. Nevertheless, the idea is that they are sufficiently accurate, but only under certain conditions.

More generally, convex relaxations can be obtained for polynomially-constrained polynomial programs, for which moment relaxation strategies have been developed [4][5]. Moment relaxations are tighter than the previously-discussed SDP relaxations, as the conventional SDP relaxation is just the *first-order* moment relaxation. Hierarchies of moment relaxations converge to the global solution.

Any SDP formulation can be relaxed further to obtain a second-order cone programming (SOCP) problem [6]. For SOCP, state-of-the-art solution methods are faster and more scalable. Furthermore, mixed-integer (MI) SOCP solvers are developed commercially whereas commercial MISDP solvers do not yet exist. Mixed-integer convex programming (MICP) refers to the optimisation of problems with integer variables, for which the continuous relaxation is convex. This generalizes MILP, MISOCP and MISDP. It is noted that any convex quadratically-constrained programming (CQCP) or convex quadratic programming (QP) problem can be reformulated as a SOCP problem. Finally, Ben-Tal and Nemirovski developed a general approach to reformulate SOCP problems as LP, to an arbitrary accuracy [7]. This lifted polyhedral relaxation technique can be used to obtain trade-offs between accuracy and speed for any SOCP problem.

To obtain rank-1 solutions where the convex relaxation solution is otherwise inexact, rank minimisation heuristics [8] were developed. The rank constraint of the original problem is thereby replaced with a penalty for its convex relaxation. In certain cases, a hidden rank-1 solution is recovered in the original (global) optimum, in others, some actual increase in the costs is observed due to the penalisation.

Solution methods

For different classes of optimisation problems, different algorithms are used. Nonconvex formulations can be solved using global solvers (e.g. spatial branch-and-bound, B&B) or local solvers (interior-point method, IPM). All convex problems can be solved to global

1 Linear programming (LP); quadratic programming (QP); second-order cone programming (SOCP); semidefinite programming (SDP), nonconvex quadratically constrained programming (NCQCP); nonlinear programming (NLP)

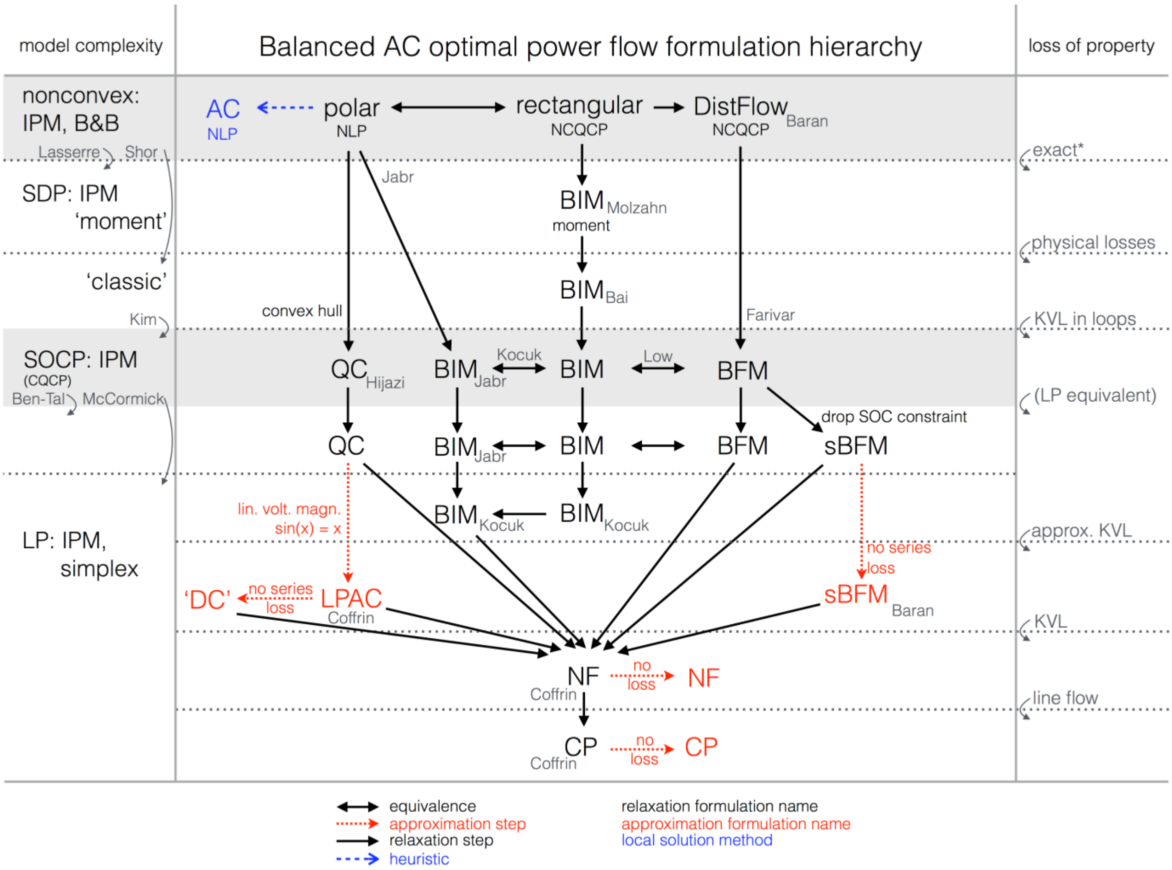

Figure 2: Relationships between AC power flow formulations, relaxations and approximations.

optimality using IPM.
For linear problems, the simplex algorithm can be used as well. In the context of mixed-integer optimisation, it may be worth the time to formulate the Ben-Tal polyhedral relaxation of a SOCP model, as then a simplex algorithm can be used – MIP simplex algorithms may offer superior performance.

OPF FORMULATION COMPARISON

Typically, the power flow equations are considered in complex form due to the compact notation, but a real-valued equivalent can be developed. This distinction is not further considered in the remainder of the article.
This paragraph details the hierarchy of convex relaxations and approximations of OPF problems. Each approximation or relaxation leads to a loss of a certain property of the initial problem, therefore, depending on the original problem, some relaxations or approximations are better suited. Furthermore, Figure 2 is used as a common thread running through the sections, and the reader is recommended to consult it readily during reading.

Nonconvex AC OPF

In local nonconvex solvers, typically the polar formulation of the OPF is used. Often the global solution is returned, which can be validated by global solvers, but only for small case studies. For larger ones, convex relaxation models can prove global optimality. Nevertheless, in certain cases these solvers converge to local solutions [9].

Convex BFM & BIM formulations

The power flow equations are easily written as a set of nonconvex quadratic equations, therefore the SDP relaxation strategy can also be followed, as set out in the previous section. One thereby obtains the SDP relaxation of OPF [10].
Two main formulations of SOCP convex relaxations of the power flow equations have been of interest in the scientific literature: the (admittance-based) bus injection model (BIM) [11] and the (impedance-based) branch flow model (BFM) [2] formulation. These relaxations have been shown to be equivalent for radial grids [12]. Both approaches obtain the equivalent solution set, albeit in different variables. The Quadratic Convex or QC formulation, is based on a different relaxation process [13]. The main relaxation steps are:

- convex hulls of sine function and cosine function;
- convex hull of quadratic terms;
- convex hull of multiplication through

McCormick's envelopes.

The main loss of property in the BIM or BFM SOCP relaxations is that Kirchoff's Voltage Law (KVL) in loops is not guaranteed. Resulting from this is that the BIM or BFM SOCP relaxation is exact under mild conditions in radial grids. It can be shown that for radial grids the SDP relaxation and the second-order cone programming (SOCP) relaxation are equivalent. When inexact, the obtained formulation may require a heuristic. In difficult situations, e.g. in overvoltage situations, finding a suitable heuristic may be a complex process.

The SOCP constraints can be relaxed further to LP complex formulations, using the Ben-Tal polyhedral relaxation technique. Still, the BIM or BFM SOCP relaxation dominates its Ben-Tal polyhedral relaxation.

Rank minimisation heuristics

The application of a rank minimisation heuristic is illustrated by [15] for the balanced BIM SDP formulation. However, there is no reason that rank minimisation strategies cannot be applied to SOCP problems in radial grids (or their polyhedral relaxation equivalents), as any true SOCP constraint is equivalent to positive semidefinite condition on a 2x2 matrix. Strangely, rank-minimisation strategies in radial grids (SOCP BIM/BFM) have not been a topic of research to the authors' knowledge, even though they could extend the scope of such approaches where they would otherwise be inexact (e.g. overvoltage).

Other compatible approaches

If the focus is to deal with meshed grids, tighter relaxations can be considered. For instance, by combining QC and BIM SDP.

In addition to the discussed relaxations, valid (convex) inequalities can be added to the formulations, to obtain an overall tighter relaxation. Such constraints are not always an immediate result of the relaxation steps. Examples are the inclusion of convex arctangent envelopes in the voltage angle equations [14].

Other approaches are detailed in the literature. Examples are chordal extensions, and cuts to separate the SOCP relaxation from SDP [14].

Approximation strategies

In addition to relaxation strategies, an approximation strategy can be followed to improve computational tractability of the original OPF problem. Depending on the characteristics of the original problem, some approximations are more accurate than others.

Approximations can be derived from different OPF formulations:

LPAC: derived from polar / QC

Although the QC formulation was proposed after the LPAC formulation, LPAC can be seen as an approximation of QC. The LPAC [16] formulation is based on four approximation steps:

- voltage magnitude values are initialized using a power flow solver;
- the relationship between reactive power flows and voltage magnitudes is linearized with respect to these initial values;
- sine function is linearized ($\sin(\theta) = \theta$);
- cosine function, as in the QC formulation, is modelled through a convex hull.

DC: derived from polar / QC

Furthermore, the linearized 'DC' formulation can be seen as a next-step approximation of LPAC, adding the following approximations:

- active power series losses are negligible;
- $\cos(\theta)$ is approximately 1;
- voltages are approximately 1 pu.

Typically, the reactive power flows in a DC OPF can be dealt with as follows:

- neglect reactive power flow;
- include reactive power flow as NF formulation;
- include the 'decoupled' reactive power flow formulation (linearization of reactive vs. voltage magnitude).

Simplified DistFlow: derived from BFM

The simplified DistFlow formulation is based on the following (equivalent) approximation steps with respect to BFM:

- the line series losses are negligible and set to zero (neglecting series current magnitude);
- the voltage drop is linearized with respect to active and reactive power flow.

In a more general case, the line losses could be approximated based on an initial power flow solution.

Network flow and copper plate

Generally, network flow (NF) and copper plate (CP) formulations have historically been considered as approximations or strong simplifications of ACOPF, however, they can be derived as relaxations as well [17]. Relative to the relaxation formulations, the classic NF or CP *approximations* take the following step:

- power flow is lossless.

Similar to the 'DC' OPF, the NF and CP approximations can opt to fully neglect reactive power flow.

Shunt elements

As noted by [18], it is important to include shunt contributions to the power flow approximations, as such error is generally cumulative. For instance, even for the classic lossless 'DC' formulation, as 'lossless' generally only refers to the *series* flow through the pi-section.

Combining formulations

In different parts of a power system, different formulations can be used. E.g., in meshed grids, the linearised 'DC' can be used; in a radial zone, the SOCP BFM. To combine them, a nodal power balance is used. Furthermore, lossless network flow can be used to bypass the numerical issues related to very low impedance values for short lines or circuit breakers.

CONCLUSION

This paper provides an overview of different OPF formulations for inclusion in market clearing algorithms. Due to the fact that integer variables are generally required in market clearing constraints, the approach to model the power flow physics for such problems needs to be tractable in combination with integer variables. In that sense, mixed-integer nonconvex optimisation is not a suitable approach. The relaxation and approximations discussed in the work allow to develop an overall model which is mixed-integer convex, which has superior tractability.

The SOCP BFM offers both high accuracy and computational tractability, but may require a heuristic when inexact. In realistic cases, optimality and feasibility of this formulation is equal to the original nonconvex problem. Furthermore, in some cases, the SOCP solution (global optimum) is actually superior to those obtained by local nonconvex solvers. Nevertheless, the heuristic can be cumbersome in difficult situations such as cases with overvoltage.

The simplified DistFlow formulation offers a high-quality approximation in terms of feasibility and optimality for distribution grids. The major advantages are that no heuristic fine tuning is required in this method and that the problem formulation is immediately linear. It is noted that simplified DistFlow for radial grids is the natural equivalent of linearized 'DC' OPF for meshed grids.

Applying the Ben-Tal reformulation technique selectively to the SOCP constraints is a path for fine-tuning accuracy selectively. Furthermore, this reformulation technique is an interesting path to leverage warm-start capability of the simplex algorithm.

For the SmartNet project market clearing simulation, a combination of 'DC' for the meshed system parts with convexified or simplified DistFlow is targeted.

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